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On the growth, limiting thickness and dominant eddy scale of turbulent shearing layers in the atmosphere

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The growth of an unbounded, density-stratified, turbulent shearing layer in the presence of a gravity field is studied using the postulate of marginal instability. It is found, for a similarity mean velocity and density distribution, that after a rapid initial growth rate the growth slows asymptotically to zero as the Richardson number approaches a value of $\frac{1}{4}$. Furthermore, the theory predicts a constant dominant turbulent eddy scale in all but the initial stages of growth of the turbulent shearing layer.

Both the general growth characteristics and the constant dominant turbulent eddy scale predicted by the theory are confirmed by experimental data.

1. Introduction

The stability of an unbounded laminar shearing layer with density stratification in the presence of a gravity field has been studied by many authors including Helmholtz (1882), Kelvin (1871), Taylor (1931), Drazin (1958), Miles (1961) and Howard (1963). The effect of viscosity on the stability of such a flow was first investigated by Koppel (1964) and solutions of the unbounded problem obtained by Maslowe & Thompson (1971).

The principle of marginal instability was used by Lessen & Singh (1974) and Lessen & Paillet (1976) to predict the spatial spread of unbounded turbulent shearing layers, jets and wakes along with the dominant eddy scales of such flows. This principle will now be applied to study the timewise growth, limiting thickness and dominant eddy scale of a turbulent shearing layer in the atmosphere using Maslowe & Thompson's calculated linear stability results for the laminar case.

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2. Theoretical considerations

Horizontal parallel flow of an incompressible fluid whose density is a function of height may be modelled, in the Boussinesq approximation, in the manner of Koppel (1964) as

$$\frac{\partial u}{\partial t} = \nu\left(\frac{\partial^2 u}{\partial y^2}\right), \quad \frac{\partial T}{\partial t} = \frac{\kappa}{\rho C}\left(\frac{\partial^2 T}{\partial y^2}\right), \quad \rho = \rho_0[1-\hat{\beta}(T-T_0)], \tag{1}$$

where t is time, y is the vertical co-ordinate (positive upwards), u is the horizontal velocity, T is the temperature, ν is the kinematic viscosity (independent of y), κ is the thermal conductivity, $\hat{\beta}$ is the thermal coefficient of expansion, C is the specific heat, ρ is the density and the subscript zero denotes a reference quantity.

For the case where ν varies with time, (1) may be solved as follows. Define τ, η and $\hat{g}$ such that

$$\tau = \int_0^t \nu(t')\,dt', \quad \eta = y\tau^{-\frac{1}{2}}, \tag{2}$$

$$u(y,t) = U_0\,\hat{g}(\eta). \tag{3}$$

Then

$$\frac{\partial u}{\partial t} = -\frac{\nu y}{2\tau^{\frac{3}{2}}}\hat{g}', \quad \frac{\partial^2 u}{\partial y^2} = \frac{1}{\tau}\hat{g}'',$$

where a prime indicates $d/d\eta$. Substituting these equations into (1) yields

$$-\tfrac{1}{2}\eta\hat{g}' = \hat{g}'' \tag{4}$$

and therefore

$$u = U_0 \operatorname{erf}\left\{y\Big/2\left[\int_0^t \nu\,dt'\right]^{\frac{1}{2}}\right\}. \tag{5}$$

In like manner

$$T = T_0 + \hat{T}\operatorname{erf}\left\{Pr^{\frac{1}{2}}y\Big/2\left[\int_0^t \nu\,dt'\right]^{\frac{1}{2}}\right\}, \tag{6}$$

$$\rho = \rho_0 - \hat{\rho}\operatorname{erf}\left\{Pr^{\frac{1}{2}}y\Big/2\left[\int_0^t \nu\,dt'\right]^{\frac{1}{2}}\right\}, \tag{7}$$

where the Prandtl number $Pr = \rho\nu C/\kappa$.

Browning & Watkins (1970) and Browning (1971), in their observations of Kelvin–Helmholtz billows in the atmosphere, noted that the thickness of the shearing layer corresponded closely to the thickness of the region of rapid temperature variation; it therefore seems reasonable to assume $Pr = 1$ for the turbulent case in the atmosphere. The same observation seems to be only approximately justified by the experimental study of Koop (1976) concerning instability and turbulence in a salinity-stratified shearing layer in water.

The case $Pr \neq 1$ is of interest because the velocity and density distributions may have different characteristic length scales. Since turbulent mass transport in the shearing layer occurs mainly within the wave breaking zone whereas turbulent momentum transport occurs via the Reynolds stress both within and outside the wave breaking zone, the turbulent Prandtl number is greater than unity. However, for the present, considerations will be limited to $Pr = 1$. Since (5) is a similarity solution for the shearing layer, the shearing-layer thickness δ may be defined as

$$\delta = \left[\int_0^t \nu(t')\,dt'\right]^{\frac{1}{2}},$$

or

$$d\delta/dt = \nu/2\delta. \tag{8}$$

Maslowe & Thompson (1971) studied the stability of a steady, incompressible, viscous shearing layer for dimensionless velocity and density profiles given by

$$\bar{u} = \tanh y^*, \quad \bar{\rho} = \exp(-\beta \tanh y^*), \tag{9}$$

where $\bar{u}$ and $\bar{\rho}$ are the dimensionless velocity and density respectively, y^* is the dimensionless vertical co-ordinate and δ^* is the associated length scale. Although the profiles given by (9) differ in a minor way from the similarity solutions given in (5)–(7), the stability characteristics calculated by Maslowe & Thompson (1971) will still apply, to a good approximation, provided that we express all dependent functions in terms of the similarity variable η. It is significant that the Maslowe & Thompson (1971) investigation dealt with a basic flow that had a Prandtl number of unity while the perturbation equations were studied for a Prandtl number of 0·72. It is clear that the stability properties of the flow were probably not greatly affected by the Prandtl number used in the perturbation equations. A basic flow with proper Prandtl number dependence would have made for a more consistent investigation.

From the properties of the velocity distributions given in (5) and (9), it can be seen that for the maximum dimensional velocity gradients to be the same it is necessary that

$$\delta = \tfrac{1}{2}\delta^* \sqrt{\pi}. \tag{9a}$$

The minimum critical Reynolds numbers R_c and the corresponding dimensionless disturbance wavenumbers α_c obtained by Maslowe & Thompson (1971) for a range of Richardson numbers can be closely approximated from their results as

$$R_c = 6J_0/(\tfrac{1}{4} - J_0), \quad \alpha_c = 2J_0, \tag{10}$$

where $J_0 = g\beta\delta^*/U_0^2$ is the Richardson number and $R_c = U_0 \delta^*/\nu$ is the Reynolds number.

Since in the case of a turbulent shearing layer the mean dimensionless velocity distribution corresponds closely to that of a laminar shearing layer, it is presumed that the same holds for the velocity and density distributions in the present case.

It is now postulated that the stratified turbulent shearing layer is marginally unstable, i.e. that the Reynolds number (based on eddy viscosity) corresponds to the minimum critical Reynolds number found for the laminar flow as in (10). Introducing (9a) and (10) into (8), it follows that

$$\frac{d\delta^*}{dt} = \frac{U_0^2}{g\beta}\frac{dJ_0}{dt} = \frac{2\nu}{\pi\delta^*} = \frac{2U_0}{\pi R_c} = \frac{U_0}{3\pi}\frac{(\frac{1}{4} - J_0)}{J_0}, \tag{10a}$$

or

$$dJ_0/d\hat{t} = (1 - 4J_0)/48J_0,$$

which yields

$$\hat{t} = -3\ln|1 - 4J_0| - 12J_0, \tag{11}$$

where

$$\hat{t} = 4g\beta t/\pi U_0.$$

The dominant eddy wavenumber α_e, which is equal to the critical dimensionless wavenumber α_c corresponding to the Reynolds number R_c, is

$$\alpha_e = 2J_0 = 2g\beta\delta^*/U_0^2.$$

The dimensional dominant eddy wavelength Λ is therefore

$$\Lambda = 2\pi\delta^*/\alpha_e = \pi U_0^2/g\beta, \tag{12}$$

and can be seen to be, within the approximation of the theory developed herein, independent of time and therefore thickness.

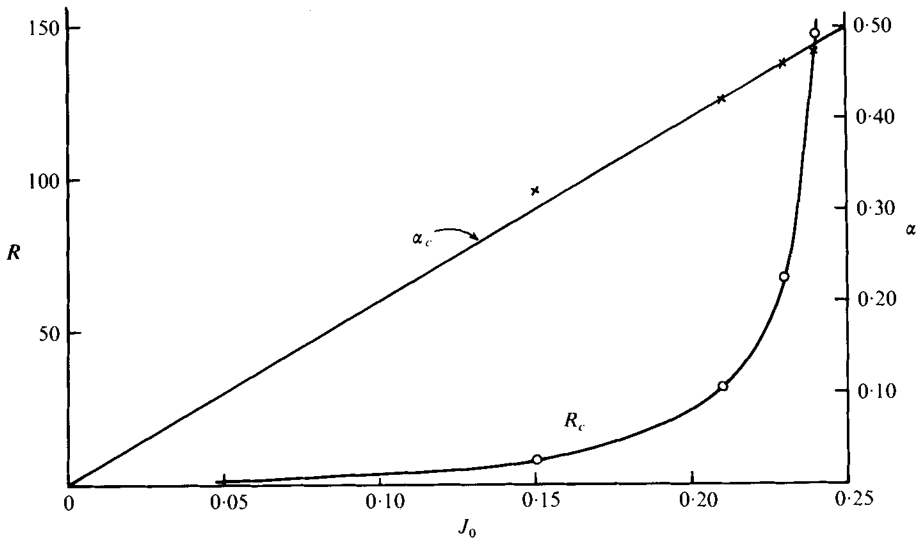

FIGURE 1. Critical Reynolds number and wavenumber *vs.* Richardson number according to (10). ×, ○, points from Maslowe & Thompson (1971).

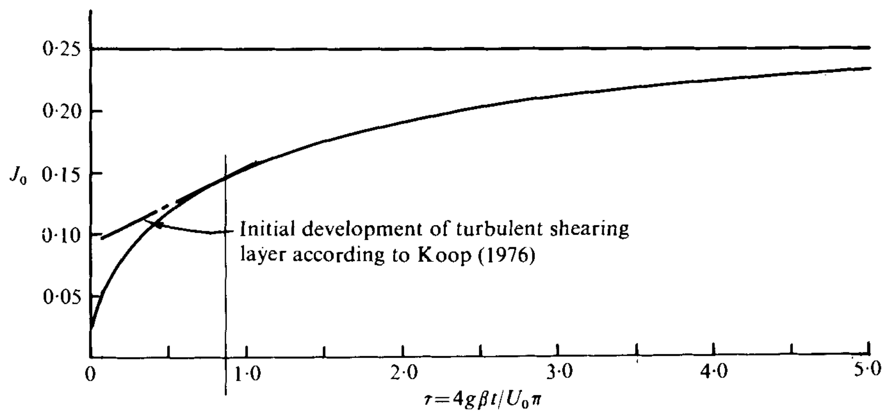

FIGURE 2. Richardson number *vs.* scaled time for a developing turbulent shearing layer.

3. Results and discussion

The data obtained from Maslowe & Thompson (1971) and the best-fit curves (10) used in the development of the theory are shown in figure 1. A plot of J_0 *vs.* $4g\beta t/\pi U_0$ for the turbulent stratified shearing layer derived from (11) is given in figure 2. It can be seen that the shearing layer thickens rapidly at first (when the critical Reynolds number based on the eddy viscosity is small) and approaches a limiting thickness asymptotically. The approximations of the model are that the mean flow is parallel, that the e-folding rate of the thickness growth is small compared with the disturbance frequency, that the molecular viscosity is negligible compared with the eddy viscosity and that the turbulent Prandtl number is unity.

The most serious of the approximations is the one which tacitly assumes that the initial rate of thickness growth is small compared with the frequency, and the poor representation of the actual phenomenon by the theory takes place when the Richardson number $J_0 \approx 0$. For this case, the Maslowe–Thompson (1971) calculations give both a minimum critical Reynolds number and a critical wavenumber of zero (the neutral-stability curve passes through the origin in the α, R plane). If the theory included a correction that accounted for the disturbance decay due to the growth rate of the thickness scale, similar to that in the calculation of the low Reynolds number stability characteristics of the half-jet by Lessen & Ko (1969), a non-zero critical Reynolds number and a non-zero critical wavenumber would occur on the neutral-stability curve for the Maslowe–Thompson (1971) problem, demonstrating relative insensitivity to changes in the Richardson number at low values of the Richardson number. The shearing layer would then (cf. the findings of Lessen & Paillet 1976) undergo a linear initial rate of growth instead of the infinite rate of growth given by the approximate theory embodied in (11); also, the dominant dimensional disturbance wavelength or turbulent eddy scale would grow linearly initially instead of remaining fixed as indicated in the approximate theory of (12). The initial rate of spread of the turbulent stratified shearing layer observed by Koop (1976) is given in his figure 42*b* and is

$$d\theta_u/dx \simeq 0{\cdot}019, \tag{13}$$

where θ_u is the 'momentum' thickness calculated from the mean velocity profile and x is the streamwise co-ordinate measured from the initial point of contact of the mixing streams. For a (spatially) uniform mixing–shearing layer, the equivalent time co-ordinate is given by

$$t = x/U_{\mathrm{av}}, \tag{14}$$

where U_{av} is the mean of the velocities of the mixing streams. To translate (13) into the notation used here, we write

$$d\theta_u/dt = U_{\mathrm{av}}\, d\theta_u/dx. \tag{15}$$

Since $\hat{t} = 4g\beta t/\pi U_0$ and θ_u as defined in Koop (1976) is $\frac{1}{2}\delta^*$, (15) finally becomes

$$\frac{dJ_0}{d\hat{t}} = \left(\frac{\pi}{2}\right)\frac{U_{\mathrm{av}}}{U_0}\frac{d\theta_u}{dx}. \tag{16}$$

Again from Koop (1976, table I), $U_0 = 3{\cdot}9$ cm s^{-1} and $U_{\mathrm{av}} = 7{\cdot}1$ cm s^{-1}, yielding $dJ_0/d\hat{t} \simeq 0{\cdot}054$, which, from (10*a*) and (11), occurs when $J_0 \simeq 0{\cdot}15$ and $\hat{t} \simeq 0{\cdot}85$. From (10) at $J_0 = 0{\cdot}15$, $R_c = 9$, which is comparable to the minimum critical Reynolds number of 12 found with the same length and velocity scales for a homogeneous half-jet corrected for small Reynolds number by Lessen & Ko (1969). The half-jet results apply only qualitatively since the velocity profile is given by the Blasius equation instead of (9). After the initial period of growth, when the Maslowe–Thompson (1971) theory more accurately models the stability of the flow, the behaviour of (11) and (12) is substantiated by the experimental data of Koop (1976), which indicate that in the later development of the turbulent shearing layer 'vortex pairing' ceases and finally the vortices themselves 'collapse' or die out (the terms in inverted commas are Koop's).

Finally, it is worthwhile to remark on the effects of non-similar growth of the shearing layer and turbulent Prandtl numbers greater than one. Because of these effects, the nominal Richardson number may grow to exceed 0·25 in a layer bounded above and below by two regions of local Richardson number less than 0·25, in which case the initial turbulent shearing layer may split into two turbulent layers. This phenomenon has also been observed by Browning & Watkins (1970) and Browning (1971). Theoretical justification for the bifurcation to two critical layers is given in Hazel (1972).

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